$\Rightarrow$ I_n is identity element of G.

(iv) Existence of Inverse: let $A \in G$ then $\det A = 1$

$\Rightarrow \det A^{-1} = \frac{1}{\det A} = \frac{1}{1} = 1$

$\Rightarrow A^{-1} \in G$.

Also $AA^{-1} = I_n = A^{-1}A$

Hence inverse of every element of A exists.

Hence G is a group under matrix multiplication.

This group is known as Special linear group of degree n and denoted by $SL(n, \mathbb{R})$.

7) Consider the collection of Complex numbers $\mathbb{C}$. Then $\mathbb{C}$ is a group under addition.

Solution: (i) closure law:

let $a+ib,\ c+id \in \mathbb{C}$, where $a, b, c, d \in \mathbb{R}$.

then $(a+ib) + (c+id) = (a+c) + i(b+d) \in \mathbb{C}$.

$\therefore$ $\mathbb{C}$ is closed under '+'.

(ii) Associative law:

let $a+ib,\ c+id,\ e+if \in \mathbb{C}$ then

$(a+ib) + (c+id + e+if) = (a+ib) + ((c+e) + i(d+f))$

$= a+(c+e) + i\ b+(d+f)$

$= (a+c)+e + i(b+d)+f$, since associativity holds in $\mathbb{R}$.

$= ((a+c) + i(b+d)) + (e+if)$

$= ((a+ib) + (c+id)) + (e+if)$

Hence associativity holds in $\mathbb{C}$.

(iii) Existence of identity:

Consider $0+i0 \in \mathbb{C}$

then for any element $(a+ib) + (0+i0)$

$= (a+0) + i(b+0)$

$= a+ib$

and $(0+i0)+(a+ib)$

$= (0+a)+i(0+b)$

$= a+ib.$

$\therefore$ $0+i0$ is identity element of $\mathbb{C}$.

(iv) Existance of Inverse: let $a+ib \in \mathbb{C}$ then $(-a)+i(-b) \in \mathbb{C}$

Such that

$$(a+ib) + (-a)+i(-b) = (a-a)+i(b-b)$$
$$= 0+i0$$

and

$$(-a)+i(-b) + a+ib = (-a+a)+i(-b+b)$$
$$= 0+i0$$

$\therefore$ inverse of every element in $\mathbb{C}$ exists.

(v) Abelian: for any ~~el~~ two elements $a+ib, c+id \in \mathbb{C}$, we have:

$$(a+ib)+(c+id) = (a+c)+i(b+d)$$
$$= (c+a)+i(d+b),$$

since Commutativity holds in $\mathbb{R}$

$$= (c+id)+(a+ib)$$

$\therefore$ $\mathbb{C}$ is abelian.

Hence all the properties of a group are satisfied by $\mathbb{C}$ and so $\mathbb{C}$ is a group under addition.

10.) $\mathbb{C}^*$ = Collection of all non-zero Complex numbers form a group under multiplication.

Solution: (i) Closure law:

let $a+ib, c+id \in \mathbb{C}$ then

$$(a+ib).(c+id) = (ac-bd)+i(bc+ad) \in \mathbb{C}.$$

$\therefore$ $\mathbb{C}^*$ is closed under '.'.

(ii) Associative law! let $(a+ib)$, $c+id$, $e+if \in \mathbb{C}$ then

$$[(a+ib)(c+id)].(e+if) = [(ac-bd)+i(ad+bc)].(e+if)$$

$$= [(ac-bd)e - (ad+bc)f] + i[(ad+bc).e + (ac-bd)f]$$

$$= [a(ce-df) - b(de+cf)] + i[a(de+cf) + b(ce-df)]$$

$$= (a+ib).[(c+id).(e+if)]$$

$\therefore$ Associative law holds in $\mathbb{C}$.

(iii) Existence of identity: let $1+i0 \in \mathbb{C}^*$ then

$$(a+ib).(1+i0) = a+ib$$

$$= (1+i0).(a+ib)$$

$\therefore$ $(1+i0)$ is identity element of $\mathbb{C}$.

(iv) Existence of inverse: let $a+ib \in \mathbb{C}^*$ then $a+ib \neq 0$

$\Rightarrow$ $\dfrac{1}{a+ib}$ is defined

and $\dfrac{1}{a+ib} = \dfrac{a-ib}{(a+ib)(a-ib)}$

$$= \frac{a-ib}{a^2+b^2}$$

$$= \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2} \in \mathbb{C}.$$

$\therefore$ $\dfrac{a}{a^2+b^2} - i\dfrac{b}{a^2+b^2}$ is inverse of $a+ib$.

$\therefore$ inverse of every element in $\mathbb{C}^*$ exist.

(V) Abelian: for any two elements $a+ib$, $c+id \in \mathbb{C}^*$, we have

$$(a+ib).(c+id) = (ac-bd) + i(bc+ad)$$

$$= (ca-db) + i(cb+da)$$

$= (t+id)\cdot(a+ib)$

$\therefore$ Commutative property holds in $\mathbb{C}$.

Hence $\mathbb{C}^*$ satisfies all the properties of a group and so $\mathbb{C}^*$ is a group under multiplication.

11.) Let X be any non-empty set and $P(X)$ denotes the power set of X. Then

(a) $P(X)$ is a monoid under the binary operation $\cap$, intersection of sets.

(b) $P(X)$ is a monoid under the binary operation $\cup$, union of sets.

Solution: (a) (i) Closure law:

Let $A, B \in P(X)$

then $A \cup B$, $A \cap B \in P(X)$

$\Rightarrow$ $P(X)$ is closed under '$\cup$' as well as '$\cap$'.

(ii) Associative law:

Let $A, B, C \in P(X)$ then

$(A \cup B) \cup C = A \cup (B \cup C)$ and

$(A \cap B) \cap C = A \cap (B \cap C)$ [By set theoratic properties]

$\therefore$ associative law holds in $P(X)$ under '$\cup$' and '$\cap$'.

(iii) Existence of Identity:

Since $\phi \in P(X)$ and $X \in P(X)$ such that

$A \cup \phi = A \quad \forall A \in P(X)$

and $A \cap X = A \quad \forall A \in P(X)$

$\Rightarrow$ ϕ is identity element of $P(X)$ under '$\cup$'.

and X is identity element of $P(X)$ under $\cap$.

Hence $P(X)$ is a monoid under '$\cup$' and '$\cap$'.